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Common Fixed Point Theorems for Four maps Using Generalized Altering Distance Functions in Seven Variables and Applications to Integral Type Inequalities

K. Sridevi[1], M.V.R. Kameswari[2], D.M.K Kiran[3]
[1]Asst. Professor of Mathematics, Dr. B. R. Ambedkar Open University, Road No. – 46, Jubilee Hills, Hyderabad –
[2]Associate Professor, Department of Engineering Mathematics, GIT, Gitam University, Visakhapatnam-530 045, India
[3]Asst. Professor in Mathematics, Department of Mathemaics, Vizag Institute of Technology, Affiliated to JNTUK, Visakhapatnam – 531162, India
(Corresponding author: D.M.K. KIRAN)

Abstract – Proving existence and uniqueness of fixed points by using generalized altering distance function in complete metric space has nice application. In this paper, we obtain unique common fixed point results for four self mappings by altering distances in seven variables and application to integral type inequalities.

Keywords: Altering distance functions, Sub compatible, generalized altering distance, Integral type inequalities.

2010 MSC: 47H10, 54E35

1. Introduction:

Fixed point theory plays an important role in functional analysis. Fixed point theory beginning from Banach contraction principle of Banach [2] (1922) in complete metric spaces has wider applications in differential and integral equations in mathematical science and engineering. Many authors have extended, generalized and improved Banach's fixed point theorem in different ways and different generalized metric spaces.

The fixed point theorems related to altering distances between points in complete metric spaces have been obtained initially by D. Delbosco [9] and F. Skof [20] in 1977. M. S. Khan et al. [12] initiated the idea of obtaining fixed point of self maps of a metric space by altering distance between the points with the use of a certain continuous control function. K. P. R. Sastry and G. V. R. Babu [18] discussed and established the existence of fixed points for the orbits of single self-maps and pairs of self-maps by using a control function. K. P. R. Sastry et al. [17, 19] proved fixed point theorems in complete metric spaces by using a continuous control function. B. S. Choudhury et al. [7, 8], G. V. R. Babu et al. [3, 4, 5, 6], S. V. R. Naidu [13, 14], K. P. R. Rao et al. [15, 16] proved some common fixed point results by altering distances.

Aliouche [1] proved common fixed point results in symmetric spaces for weakly compatible mappings under contractive condition of integral type. Hesseni [10, 11] used contractive rule of integral type by altering distance and generalized common fixed point results. Mishra et al. [22] proved two common fixed point theorems under contraction rule of integral type in complete metric spaces by altering distance.

The main aim of this paper is to prove the existence and uniqueness of common fixed points of two pairs of sub compatible mappings by using a generalized altering distance function of seven variables and apply them to integral type inequalities. This paper is an extension of our previous result K. Sridevi et al. [21].

2. Preliminaries:

2.1 Definition [15]: A function $\psi : [0,\infty) \longrightarrow [0,\infty)$ is called an altering distance function, if it satisfies following conditions.

(1) $\psi(t)$ is monotonically increasing and continuous.

(2) $\psi(t) = 0 \quad iff \quad t = 0.$

2.2 Definition: A function $\psi : R^{+n} \longrightarrow R^{+} = [0,\infty)$ is called a generalized altering distance function on R^{+n} if ψ is continuous, monotone increasing in each variable and

$$\psi(x_1, x_2, \dots, x_n) = 0 \quad if\ and\ only\ if \quad x_1 = x_2 = \cdots = x_n = 0.$$

The collection of all generalized altering distance function on R^{+n} is denoted by Ψ_n.

Suppose $\psi \in \Psi_n$.

Now we define a function $\varphi_\psi(y)$ by $\varphi_\psi(y) = \psi(y, y, \dots, y)$ for $y \in [0,\infty)$,

Clearly $\varphi_\psi(y) = 0 \ \ if\ and\ only\ if \ \ y = 0.$

2.3 Definition: Two maps $p, q : X \longrightarrow X$ of a metric space (X, d) are called sub compatible if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} p(x_n) = \lim_{n\to\infty} q(x_n) = t,\ t \in X \ \ then \ \lim_{n\to\infty} d(pqx_n, qpx_n) = 0.$$

3. Main Result:

3.1 Theorem: Let (X, d) be complete metric space and $f, g, U\ and\ V$ be four mappings from X to itself such that

$$\begin{aligned}\varphi_1\big(d(fx, gy)\big) \ \le\ & \psi_1\Big(d(Ux, Vy), \tfrac{1}{2}d(Ux, gy), \tfrac{1}{2}d(fx, Vy), d(Ux, fx), d(Vy, gy), \tfrac{1}{2}[d(gy, Ux) \\ & + d(fx, Vy)], \tfrac{1}{2}[d(Ux, Vy) + d(Ux, fx)]\Big) \\ & - \psi_2\Big(d(Ux, Vy), \tfrac{1}{2}d(Ux, gy), \tfrac{1}{2}d(fx, Vy), d(Ux, fx), d(Vy, gy), \tfrac{1}{2}[d(gy, Ux) \\ & + d(fx, Vy)], \tfrac{1}{2}[d(Ux, Vy) + d(Ux, fx)]\Big) - - - (3.1.1)\end{aligned}$$

for all $x, y \in X$, where $\psi_1, \psi_2 \in \Psi_7$ with $\varphi_1(\alpha) = \psi_1(\alpha, \alpha, \alpha, \alpha, \alpha, \alpha, \alpha) \ for \ \alpha \in [0,\infty)$.

i) One of the four mappings $f, g, U\ and\ V$ is continuous.

ii) $(f, U)\ and\ (g, V)$ are sub compatible.

iii) $f(X) \subseteq V(X)\ and\ g(X) \subseteq U(X)$.

Then $f, g, U\ and\ V$ have a unique common fixed point in X.

Proof: Let$x_0 \in X$. There exist $x_1\ and\ x_2$ such that

$fx_0 = Vx_1$

$gx_1 = Ux_2 \dots$

In general,

$fx_{2n} = Vx_{2n+1} \ \ and \ \ gx_{2n+1} = Ux_{2n+2}$

Substituting $x = x_{2n} \ \ and \ \ y = x_{2n+1} \ \ in \ \ (3.1.1)$, we get

$$\varphi_1\big(d(fx_{2n},gx_{2n+1})\big)$$

$$\leq \psi_1\begin{pmatrix} d(Ux_{2n},Vx_{2n+1}),\frac{1}{2}d(Ux_{2n},gx_{2n+1}),\frac{1}{2}d(fx_{2n},Vx_{2n+1}),d(Ux_{2n},fx_{2n}),d(Vx_{2n+1},gx_{2n+1}),\\ \frac{1}{2}[d(gx_{2n+1},Ux_{2n})+d(fx_{2n},Vx_{2n+1})],\frac{1}{2}[d(Ux_{2n},Vx_{2n+1})+d(Ux_{2n},fx_{2n})]\end{pmatrix}$$

$$-\psi_2\begin{pmatrix} d(Ux_{2n},Vx_{2n+1}),\frac{1}{2}d(Ux_{2n},gx_{2n+1}),\frac{1}{2}d(fx_{2n},Vx_{2n+1}),d(Ux_{2n},fx_{2n}),d(Vx_{2n+1},gx_{2n+1}),\\ \frac{1}{2}[d(gx_{2n+1},Ux_{2n})+d(fx_{2n},Vx_{2n+1})],\frac{1}{2}[d(Ux_{2n},Vx_{2n+1})+d(Ux_{2n},fx_{2n})]\end{pmatrix}$$

$$=\psi_1\begin{pmatrix} d(gx_{2n-1},fx_{2n}),\frac{1}{2}d(gx_{2n-1},gx_{2n+1}),\frac{1}{2}d(fx_{2n},fx_{2n}),d(gx_{2n-1},fx_{2n}),d(fx_{2n},gx_{2n+1}),\\ \frac{1}{2}[d(gx_{2n+1},gx_{2n-1})+d(fx_{2n},fx_{2n})],\frac{1}{2}[d(gx_{2n-1},fx_{2n})+d(gx_{2n-1},fx_{2n})]\end{pmatrix}$$

$$-\psi_2\begin{pmatrix} d(gx_{2n-1},fx_{2n}),\frac{1}{2}d(gx_{2n-1},gx_{2n+1}),\frac{1}{2}d(fx_{2n},fx_{2n}),d(gx_{2n-1},fx_{2n}),d(fx_{2n},gx_{2n+1}),\\ \frac{1}{2}[d(gx_{2n+1},gx_{2n-1})+d(fx_{2n},fx_{2n})],\frac{1}{2}[d(gx_{2n-1},fx_{2n})+d(gx_{2n-1},fx_{2n})]\end{pmatrix}$$

$$=\psi_1\begin{pmatrix} d(gx_{2n-1},fx_{2n}),\frac{1}{2}d(gx_{2n-1},gx_{2n+1}),0,d(gx_{2n-1},fx_{2n}),\\ d(fx_{2n},gx_{2n+1}),\frac{1}{2}d(gx_{2n+1},gx_{2n-1}),d(gx_{2n-1},fx_{2n})\end{pmatrix}$$

$$-\psi_2\begin{pmatrix} d(gx_{2n-1},fx_{2n}),\frac{1}{2}d(gx_{2n-1},gx_{2n+1}),0,d(gx_{2n-1},fx_{2n}),\\ d(fx_{2n},gx_{2n+1}),\frac{1}{2}d(gx_{2n+1},gx_{2n-1}),d(gx_{2n-1},fx_{2n})\end{pmatrix} ---(3.1.2)$$

Let $\alpha=\max\{d(gx_{2n-1},fx_{2n}),d(gx_{2n-1},gx_{2n+1}),d(fx_{2n},gx_{2n+1})\}$ $---(3.1.3)$

$$\varphi_1\big(d(fx_{2n},gx_{2n+1})\big)$$

$$\leq \psi_1(\alpha,\alpha,\alpha,\alpha,\alpha,\alpha,\alpha)$$

$$-\psi_2\begin{pmatrix} d(gx_{2n-1},fx_{2n}),\frac{1}{2}d(gx_{2n-1},gx_{2n+1}),0,d(gx_{2n-1},fx_{2n}),\\ d(fx_{2n},gx_{2n+1}),\frac{1}{2}d(gx_{2n+1},gx_{2n-1}),d(gx_{2n-1},fx_{2n})\end{pmatrix}$$

From (3.1.2) *and* (3.1.3), *we get*

$$\varphi_1\big(d(fx_{2n},gx_{2n+1})\big)\leq\varphi_1(\alpha)-\psi_2\begin{pmatrix} d(gx_{2n-1},fx_{2n}),\frac{1}{2}d(gx_{2n-1},gx_{2n+1}),0,d(gx_{2n-1},fx_{2n}),\\ d(fx_{2n},gx_{2n+1}),\frac{1}{2}d(gx_{2n+1},gx_{2n-1}),d(gx_{2n-1},fx_{2n})\end{pmatrix}$$

From (2.2.2), *we get*

$$\varphi_1\big(d(fx_{2n},gx_{2n+1})\big)\leq\varphi_1(\alpha)\quad if\ \alpha>0,\quad \alpha=\max\{d(gx_{2n-1},fx_{2n}),d(gx_{2n-1},gx_{2n+1})\}$$

$$d(fx_{2n},gx_{2n+1})<\max\{d(gx_{2n-1},fx_{2n}),d(gx_{2n-1},gx_{2n+1})\}$$

$$d(fx_{2n},gx_{2n+1})\leq d(gx_{2n-1},fx_{2n})+d(fx_{2n},gx_{2n-1})$$

$$d(fx_{2n},gx_{2n+1})\leq d(gx_{2n-1},fx_{2n}) ----(3.1.4)$$

Substituting $x=x_{2n+2}$ *and* $y=x_{2n+1}$ *in* (3.1.1), we get

$$\varphi_1\big(d(fx_{2n+2}, gx_{2n+1})\big)$$

$$\leq \psi_1\begin{pmatrix} d(Ux_{2n+2}, Vx_{2n+1}), \frac{1}{2}d(Ux_{2n+2}, gx_{2n+1}), \frac{1}{2}d(fx_{2n+2}, Vx_{2n+1}), d(Ux_{2n+2}, fx_{2n+2}), d(Vx_{2n+1}, gx_{2n+1}), \\ \frac{1}{2}[d(gx_{2n+1}, Ux_{2n+2}) + d(fx_{2n+2}, Vx_{2n+1})], \frac{1}{2}[d(Ux_{2n+2}, Vx_{2n+1}) + d(Ux_{2n+2}, fx_{2n+2})] \end{pmatrix}$$

$$-\psi_2\begin{pmatrix} d(Ux_{2n+2}, Vx_{2n+1}), \frac{1}{2}d(Ux_{2n+2}, gx_{2n+1}), \frac{1}{2}d(fx_{2n+2}, Vx_{2n+1}), d(Ux_{2n+2}, fx_{2n+2}), d(Vx_{2n+1}, gx_{2n+1}), \\ \frac{1}{2}[d(gx_{2n+1}, Ux_{2n+2}) + d(fx_{2n+2}, Vx_{2n+1})], \frac{1}{2}[d(Ux_{2n+2}, Vx_{2n+1}) + d(Ux_{2n+2}, fx_{2n+2})] \end{pmatrix}$$

$$= \psi_1\begin{pmatrix} d(gx_{2n+1}, fx_{2n}), \frac{1}{2}d(gx_{2n+1}, gx_{2n+1}), \frac{1}{2}d(fx_{2n+2}, fx_{2n}), d(gx_{2n+1}, fx_{2n+2}), d(fx_{2n}, gx_{2n+1}), \\ \frac{1}{2}[d(gx_{2n+1}, gx_{2n+1}) + d(fx_{2n+2}, fx_{2n})], \frac{1}{2}[d(gx_{2n+1}, fx_{2n}) + d(gx_{2n+1}, fx_{2n+2})] \end{pmatrix}$$

$$-\psi_2\begin{pmatrix} d(gx_{2n+1}, fx_{2n}), \frac{1}{2}d(gx_{2n+1}, gx_{2n+1}), \frac{1}{2}d(fx_{2n+2}, fx_{2n}), d(gx_{2n+1}, fx_{2n+2}), d(fx_{2n}, gx_{2n+1}), \\ \frac{1}{2}[d(gx_{2n+1}, gx_{2n+1}) + d(fx_{2n+2}, fx_{2n})], \frac{1}{2}[d(gx_{2n+1}, fx_{2n}) + d(gx_{2n+1}, fx_{2n+2})] \end{pmatrix}$$

$$= \psi_1\begin{pmatrix} d(gx_{2n+1}, fx_{2n}), 0, \frac{1}{2}d(fx_{2n+2}, fx_{2n}), d(gx_{2n+1}, fx_{2n+2}), d(fx_{2n}, gx_{2n+1}), \\ d(fx_{2n+2}, fx_{2n}), \frac{1}{2}[d(gx_{2n+1}, fx_{2n}) + d(gx_{2n+1}, fx_{2n+2})] \end{pmatrix}$$

$$-\psi_2\begin{pmatrix} d(gx_{2n+1}, fx_{2n}), 0, \frac{1}{2}d(fx_{2n+2}, fx_{2n}), d(gx_{2n+1}, fx_{2n+2}), d(fx_{2n}, gx_{2n+1}), \\ d(fx_{2n+2}, fx_{2n}), \frac{1}{2}[d(gx_{2n+1}, fx_{2n}) + d(gx_{2n+1}, fx_{2n+2})] \end{pmatrix} --- (3.1.5)$$

Let $\beta = \max\{d(gx_{2n+1}, fx_{2n}), d(fx_{2n+2}, fx_{2n}), d(gx_{2n+1}, fx_{2n+2})\}$ ----(3.1.6)

$$\varphi_1\big(d(fx_{2n+2}, gx_{2n+1})\big)$$

$$\leq \psi_1(\beta, \beta, \beta, \beta, \beta, \beta, \beta)$$

$$-\psi_2\begin{pmatrix} d(gx_{2n+1}, fx_{2n}), 0, \frac{1}{2}d(fx_{2n+2}, fx_{2n}), d(gx_{2n+1}, fx_{2n+2}), d(fx_{2n}, gx_{2n+1}), \\ d(fx_{2n+2}, fx_{2n}), \frac{1}{2}[d(gx_{2n+1}, fx_{2n}) + d(gx_{2n+1}, fx_{2n+2})] \end{pmatrix}$$

$$\varphi_1\big(d(fx_{2n+2}, gx_{2n+1})\big)$$

$$\leq \varphi_1(\beta)$$

$$-\psi_2\begin{pmatrix} d(gx_{2n+1}, fx_{2n}), 0, \frac{1}{2}d(fx_{2n+2}, fx_{2n}), d(gx_{2n+1}, fx_{2n+2}), d(fx_{2n}, gx_{2n+1}), \\ d(fx_{2n+2}, fx_{2n}), \frac{1}{2}[d(gx_{2n+1}, fx_{2n}) + d(gx_{2n+1}, fx_{2n+2})] \end{pmatrix}$$

$$\varphi_1\big(d(fx_{2n+2}, gx_{2n+1})\big) \leq \varphi_1(\beta) \quad if\ \beta > 0, \quad \beta = \max\{d(gx_{2n+1}, fx_{2n}), d(fx_{2n+2}, fx_{2n})\}$$

$$d(fx_{2n+2}, gx_{2n+1}) < \max\{d(gx_{2n+1}, fx_{2n}), d(fx_{2n+2}, fx_{2n})\}$$

$$d(fx_{2n+2}, gx_{2n+1}) \leq d(fx_{2n+2}, gx_{2n+1}) + d(gx_{2n+1}, fx_{2n})$$

$$d(fx_{2n+2}, gx_{2n+1}) < d(gx_{2n+1}, fx_{2n}) ----(3.1.7)$$

From (3.1.4) *and* (3.1.7), *we get*

$$d(fx_{2n+2}, gx_{2n+1}) < d(fx_{2n}, gx_{2n+1}) < d(fx_{2n}, gx_{2n-1})$$

In general,

$$d(fx_{2n+2}, gx_{2n+1}) \le d(fx_{2n}, gx_{2n+1}) \le d(fx_{2n}, gx_{2n-1}) \le ---\le d(fx_2, gx_1) \le d(fx_0, gx_1)$$

From this it follows that the sequences

$\{d(fx_{2n}, gx_{2n-1})\}$ *and* $\{d(fx_{2n+2}, gx_{2n+1})\}$ are the both decreasing sequences, decreasing to the same limit.

Suppose, let $\gamma = \lim_{n \to \infty} d(fx_{2n+2}, gx_{2n+1}) = \lim_{n \to \infty} d(fx_{2n}, gx_{2n-1}) ----(3.1.8)$

On letting $n \to \infty$ *in* (3.1.5), we get

$\varphi_1(\gamma) \le \psi_1(\gamma, 0, 0, \gamma, \gamma, 0, \gamma) - \psi_2(\gamma, 0, 0, \gamma, \gamma, 0, \gamma)$

$\varphi_1(\gamma) \le \psi_1(\gamma, \gamma, \gamma, \gamma, \gamma, \gamma, \gamma) - \psi_2(\gamma, 0, 0, \gamma, \gamma, 0, \gamma)$

From (2.2.2), *we get*

$\varphi_1(\gamma) \le \varphi_1(\gamma) - \psi_2(\gamma, 0, 0, \gamma, \gamma, 0, \gamma)$

Therefore, $\varphi_1(\gamma) < \varphi_1(\gamma) \quad if \;\gamma > 0$

a contradiction.

Therefore, $\gamma = 0$.

Therefore, $\gamma = \lim_{n \to \infty} d(fx_{2n+2}, gx_{2n+1}) = \lim_{n \to \infty} d(fx_{2n}, gx_{2n-1}) = 0 ----(3.1.9)$

Write $y_n = \begin{cases} fx_n & if\;\; n \;\; is\; even \\ gx_n & if\;\; n\;\; is\; odd \end{cases}$

Therefore, $d(y_{2n+2}, y_{2n+1}) \le d(y_{2n}, y_{2n-1})$.

Therefore, $d(y_{n+1}, y_n) \to 0 \quad as \;\; n \to \infty \quad ----(3.1.10)$

Now, to prove the sequence $\{y_n\}$ is a Cauchy sequence in X,

it is sufficient to prove that $\{y_{2n}\}$ is a Cauchy sequence.

Suppose that $\{y_{2n}\}$ is not a Cauchy sequence.

Then there is an $\varepsilon > 0$, sequence $\{2m(k), 2n(k)\}$ with $k < m(k) < n(k)$

$$d(y_{2m(k)}, y_{2n(k)}) \ge \varepsilon \quad and \quad d(y_{2m(k)}, y_{2n(k)-2}) < \varepsilon$$

$$\begin{aligned} \varepsilon &\le d(y_{2m(k)}, y_{2n(k)}) \\ &\le d(y_{2m(k)}, y_{2n(k)-2}) + d(y_{2n(k)-2}, y_{2n(k)}) \\ &\le \varepsilon + d(y_{2n(k)-2}, y_{2n(k)}) \\ &\le \varepsilon + d(y_{2n(k)}, y_{2n(k)-1}) + d(y_{2n(k)-1}, y_{2n(k)-2}) \end{aligned}$$

Taking $n \to \infty$ in the inequality

$$\lim_{k\to\infty} d(y_{2m(k)}, y_{2n(k)}) = \varepsilon \quad ----(3.1.11)$$

Consider, $d(y_{2m(k)}, y_{2n(k)-1}) \leq d(y_{2m(k)}, x_{2n(k)}) + d(y_{2n(k)}, y_{2n(k)+1})$

$d(y_{2m(k)}, y_{2n(k)}) \leq d(y_{2m(k)}, y_{2n(k)-1}) + d(y_{2n(k)-1}, y_{2n(k)})$

On letting $k \longrightarrow \infty$ we get,

$$\lim_{k\longrightarrow\infty}\sup d(y_{2m(k)}, y_{2n(k)-1}) \leq \varepsilon \quad and \quad \lim_{k\longrightarrow\infty}\inf d(y_{2m(k)}, y_{2n(k)-1}) \geq \varepsilon$$

$$\Rightarrow \lim_{k\longrightarrow\infty} d(y_{2m(k)}, y_{2n(k)-1}) = \varepsilon --- (3.1.12)$$

Consider, $d(y_{2m(k)-1}, y_{2n(k)-1}) \leq d(y_{2m(k)-1}, y_{2m(k)}) + d(y_{2m(k)}, y_{2n(k)-1})$

$d(y_{2m(k)}, y_{2n(k)-1}) \leq d(y_{2m(k)-1}, y_{2n(k)-1}) + d(y_{2m(k)-1}, y_{2m(k)})$

On letting $k \longrightarrow \infty$ we get,

$$\lim_{k\longrightarrow\infty}\sup d(y_{2m(k)-1}, y_{2n(k)-1}) \leq \varepsilon \quad and \quad \lim_{k\longrightarrow\infty}\inf d(y_{2m(k)-1}, y_{2n(k)-1}) \geq \boldsymbol{\varepsilon}$$

$$\Rightarrow \lim_{k\longrightarrow\infty} d(y_{2m(k)-1}, y_{2n(k)-1}) = \varepsilon --(3.1.13)$$

Consider, $d(y_{2m(k)-1}, y_{2n(k)-2}) \leq d(y_{2m(k)-1}, y_{2n(k)-1}) + d(y_{2n(k)-1}, y_{2n(k)-2})$

$d(y_{2m(k)-1}, y_{2n(k)-1}) \leq d(y_{2m(k)-1}, y_{2n(k)-2}) + d(y_{2n(k)-2}, y_{2n(k)-1})$

On letting $k \longrightarrow \infty$ we get,

$$\lim_{k\longrightarrow\infty}\sup d(y_{2m(k)-1}, y_{2n(k)-2}) \leq \varepsilon \quad and \quad \lim_{k\longrightarrow\infty}\inf d(y_{2m(k)-1}, y_{2n(k)-2}) \geq \varepsilon$$

$$\Rightarrow \lim_{k\longrightarrow\infty} d(y_{2m(k)-1}, y_{2n(k)-2}) = \varepsilon --(3.1.14)$$

Consider, $d(y_{2m(k)}, y_{2n(k)-2}) \leq d(y_{2m(k)}, y_{2n(k)-1}) + d(y_{2n(k)-1}, y_{2n(k)-2})$

$d(y_{2m(k)}, y_{2n(k)-1}) \leq d(y_{2m(k)}, y_{2n(k)-2}) + d(y_{2n(k)-2}, y_{2n(k)-1})$

On letting $k \longrightarrow \infty$ we get,

$$\lim_{k\longrightarrow\infty}\sup d(y_{2m(k)}, y_{2n(k)-2}) \leq \varepsilon \quad and \quad \lim_{k\longrightarrow\infty}\inf d(y_{2m(k)}, y_{2n(k)-2}) \geq \varepsilon$$

$$\Rightarrow \lim_{k\longrightarrow\infty} d(y_{2m(k)}, y_{2n(k)-2}) = \varepsilon --- (3.1.15)$$

Substituting $s = x_{2m(k)}$ *and* $t = x_{2n(k)-1}$ *in* (3.1.1), we get

$$\varphi_1\left(d(fx_{2m(k)}, gx_{2n(k)-1})\right)$$

$$\leq \psi_1\begin{pmatrix} d(Ux_{2m(k)}, Vx_{2n(k)-1}), \frac{1}{2}d(Ux_{2m(k)}, gx_{2n(k)-1}), \frac{1}{2}d(fx_{2m(k)}, Vx_{2n(k)-1}), d(Ux_{2m(k)}, fx_{2m(k)}), \\ d(Vx_{2n(k)-1}, gx_{2n(k)-1}), \frac{1}{2}[d(gx_{2n(k)-1}, Ux_{2m(k)}) + d(fx_{2m(k)}, Vx_{2n(k)-1})], \\ \frac{1}{2}[d(Ux_{2m(k)}, Vx_{2n(k)-1}) + d(Ux_{2m(k)}, fx_{2m(k)})] \end{pmatrix}$$

$$-\psi_2\begin{pmatrix} d(Ux_{2m(k)}, Vx_{2n(k)-1}), \frac{1}{2}d(Ux_{2m(k)}, gx_{2n(k)-1}), \frac{1}{2}d(fx_{2m(k)}, Vx_{2n(k)-1}), d(Ux_{2m(k)}, fx_{2m(k)}), \\ d(Vx_{2n(k)-1}, gx_{2n(k)-1}), \frac{1}{2}[d(gx_{2n(k)-1}, Ux_{2m(k)}) + d(fx_{2m(k)}, Vx_{2n(k)-1})], \\ \frac{1}{2}[d(Ux_{2m(k)}, Vx_{2n(k)-1}) + d(Ux_{2m(k)}, fx_{2m(k)})] \end{pmatrix}$$

$$= \psi_1 \begin{pmatrix} d(gx_{2m(k)-1}, fx_{2n(k)-2}), \frac{1}{2} d(gx_{2m(k)-1}, gx_{2n(k)-1}), \frac{1}{2} d(fx_{2m(k)}, fx_{2n(k)-2}), d(gx_{2m(k)-1}, fx_{2m(k)}), \\ d(fx_{2n(k)-2}, gx_{2n(k)-1}), \frac{1}{2}[d(gx_{2n(k)-1}, gx_{2m(k)-1}) + d(fx_{2m(k)}, fx_{2n(k)-2})], \\ \frac{1}{2}[d(gx_{2m(k)-1}, fx_{2n(k)-2}) + d(gx_{2m(k)-1}, fx_{2m(k)})] \end{pmatrix}$$

$$- \psi_2 \begin{pmatrix} d(gx_{2m(k)-1}, fx_{2n(k)-2}), \frac{1}{2} d(gx_{2m(k)-1}, gx_{2n(k)-1}), \frac{1}{2} d(fx_{2m(k)}, fx_{2n(k)-2}), d(gx_{2m(k)-1}, fx_{2m(k)}), \\ d(fx_{2n(k)-2}, gx_{2n(k)-1}), \frac{1}{2}[d(gx_{2n(k)-1}, gx_{2m(k)-1}) + d(fx_{2m(k)}, fx_{2n(k)-2})], \\ \frac{1}{2}[d(gx_{2m(k)-1}, fx_{2n(k)-2}) + d(gx_{2m(k)-1}, fx_{2m(k)})] \end{pmatrix}$$

$\varphi_1\left(d(y_{2m(k)}, y_{2n(k)-1})\right)$

$$\leq \psi_1 \begin{pmatrix} d(y_{2m(k)-1}, y_{2n(k)-2}), \frac{1}{2} d(y_{2m(k)-1}, y_{2n(k)-1}), \frac{1}{2} d(y_{2m(k)}, y_{2n(k)-2}), d(y_{2m(k)-1}, y_{2m(k)}), \\ d(y_{2n(k)-2}, y_{2n(k)-1}), \frac{1}{2}[d(y_{2n(k)-1}, y_{2m(k)-1}) + d(y_{2m(k)}, y_{2n(k)-2})], \\ \frac{1}{2}[d(y_{2m(k)-1}, y_{2n(k)-2}) + d(y_{2m(k)-1}, y_{2m(k)})] \end{pmatrix}$$

$$- \psi_2 \begin{pmatrix} d(y_{2m(k)-1}, y_{2n(k)-2}), \frac{1}{2} d(y_{2m(k)-1}, y_{2n(k)-1}), \frac{1}{2} d(y_{2m(k)}, y_{2n(k)-2}), d(y_{2m(k)-1}, y_{2m(k)}), \\ d(y_{2n(k)-2}, y_{2n(k)-1}), \frac{1}{2}[d(y_{2n(k)-1}, y_{2m(k)-1}) + d(y_{2m(k)}, y_{2n(k)-2})], \\ \frac{1}{2}[d(y_{2m(k)-1}, y_{2n(k)-2}) + d(y_{2m(k)-1}, y_{2m(k)})] \end{pmatrix}$$

On letting $k \longrightarrow \infty$, $from$ (3.1.11), (3.1.12), (3.1.13), (3.1.14), (3.1.15), $we\ get$

$$\varphi_1(\varepsilon) \leq \psi_1\left(\varepsilon, \frac{\varepsilon}{2}, \frac{\varepsilon}{2}, 0, 0, \varepsilon, \frac{\varepsilon}{2}\right) - \psi_2\left(\varepsilon, \frac{\varepsilon}{2}, \frac{\varepsilon}{2}, 0, 0, \varepsilon, \frac{\varepsilon}{2}\right)$$

Therefore, $\varphi_1(\varepsilon) \leq \psi_1(\varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon) - \psi_2\left(\varepsilon, \frac{\varepsilon}{2}, \frac{\varepsilon}{2}, 0, 0, \varepsilon, \frac{\varepsilon}{2}\right)$

Therefore, $\varphi_1(\varepsilon) \leq \varphi_1(\varepsilon) - \psi_2\left(\varepsilon, \frac{\varepsilon}{2}, \frac{\varepsilon}{2}, 0, 0, \varepsilon, \frac{\varepsilon}{2}\right)$

Therefore, $\varphi_1(\varepsilon) < \varphi_1(\varepsilon)$ (since $\varepsilon > 0$ and hence $\psi_2\left(\varepsilon, \frac{\varepsilon}{2}, \frac{\varepsilon}{2}, 0, 0, \varepsilon, \frac{\varepsilon}{2}\right) > 0$)

a contradiction.

Therefore, $\{y_{2n}\}$ is a Cauchy sequence.

Similarly we can show that, $\{y_{2n+1}\}$ is a Cauchy's sequence.

Since $d(y_n, y_{n+1}) \longrightarrow 0$ as $n \longrightarrow \infty$

$\{y_n\}$ is a Cauchy sequence.

Therefore there exists l such that $\{y_n\} \longrightarrow l\ as\ n \longrightarrow \infty$

Since $(f, U), (g, V)$ are sub compatible,

Suppose U is continuous function.

Then $Ux_{2n} \longrightarrow l \Longrightarrow gx_{2n-1} \longrightarrow l$ and $Ufx_{2n} \longrightarrow Ul$, $UUx_{2n} \longrightarrow Ul$ ----(3.1.16)

Since (f, U) is sub compatible, we have $fUx_{2n} \longrightarrow Ul$ $\quad ----(3.1.17)$

Substituting $\quad s = Ux_{2n} \;\; and \;\; t = x_{2n-1} \;\; in$ (3.1.1), we get

$$\varphi_1\big(d(fUx_{2n}, gx_{2n-1})\big)$$
$$\leq \psi_1\begin{pmatrix} d(UUx_{2n}, Vx_{2n-1}), \frac{1}{2}d(UUx_{2n}, gx_{2n-1}), \frac{1}{2}d(fUx_{2n}, Vx_{2n-1}), d(UUx_{2n}, fUx_{2n}), d(Vx_{2n-1}, gx_{2n-1}), \\ \frac{1}{2}[d(gx_{2n-1}, UUx_{2n}) + d(fUx_{2n}, Vx_{2n-1})], \frac{1}{2}[d(UUx_{2n}, Vx_{2n-1}) + d(UUx_{2n}, fUx_{2n})] \end{pmatrix}$$
$$-\psi_2\begin{pmatrix} d(UUx_{2n}, Vx_{2n-1}), \frac{1}{2}d(UUx_{2n}, gx_{2n-1}), \frac{1}{2}d(fUx_{2n}, Vx_{2n-1}), d(UUx_{2n}, fUx_{2n}), d(Vx_{2n-1}, gx_{2n-1}), \\ \frac{1}{2}[d(gx_{2n-1}, UUx_{2n}) + d(fUx_{2n}, Vx_{2n-1})], \frac{1}{2}[d(UUx_{2n}, Vx_{2n-1}) + d(UUx_{2n}, fUx_{2n})] \end{pmatrix}$$

$$\varphi_1\big(d(fUx_{2n}, gx_{2n-1})\big)$$
$$\leq \psi_1\begin{pmatrix} d(UUx_{2n}, fx_{2n-2}), \frac{1}{2}d(UUx_{2n}, gx_{2n-1}), \frac{1}{2}d(fUx_{2n}, fx_{2n-2}), d(UUx_{2n}, fUx_{2n}), d(fx_{2n-2}, gx_{2n-1}), \\ \frac{1}{2}[d(gx_{2n-1}, UUx_{2n}) + d(fUx_{2n}, fx_{2n-2})], \frac{1}{2}[d(UUx_{2n}, fx_{2n-2}) + d(UUx_{2n}, fUx_{2n})] \end{pmatrix}$$
$$-\psi_2\begin{pmatrix} d(UUx_{2n}, fx_{2n-2}), \frac{1}{2}d(UUx_{2n}, gx_{2n-1}), \frac{1}{2}d(fUx_{2n}, fx_{2n-2}), d(UUx_{2n}, fUx_{2n}), d(fx_{2n-2}, gx_{2n-1}), \\ \frac{1}{2}[d(gx_{2n-1}, UUx_{2n}) + d(fUx_{2n}, fx_{2n-2})], \frac{1}{2}[d(UUx_{2n}, fx_{2n-2}) + d(UUx_{2n}, fUx_{2n})] \end{pmatrix}$$

$From$ (3.1.16), (3.1.17), $we\ get$

$$\varphi_1\big(d(Ul, l)\big) \leq \psi_1\begin{pmatrix} d(Ul, l), \frac{1}{2}d(Ul, l), \frac{1}{2}d(Ul, l), d(Ul, Ul), d(l, l), \frac{1}{2}[d(l, Ul) + d(Ul, l)], \\ \frac{1}{2}[d(Ul, l) + d(Ul, Ul)] \end{pmatrix}$$
$$-\psi_2\begin{pmatrix} d(Ul, l), \frac{1}{2}d(Ul, l), \frac{1}{2}d(Ul, l), d(Ul, Ul), d(l, l), \frac{1}{2}[d(l, Ul) + d(Ul, l)], \\ \frac{1}{2}[d(Ul, l) + d(Ul, Ul)] \end{pmatrix}$$
$$= \psi_1\left(d(Ul, l), \tfrac{1}{2}d(Ul, l), \tfrac{1}{2}d(Ul, l), 0, 0, d(l, Ul), \tfrac{1}{2}d(Ul, l)\right) -$$
$$\psi_2\left(d(Ul, l), \tfrac{1}{2}d(Ul, l), \tfrac{1}{2}d(Ul, l), 0, 0, d(l, Ul), \tfrac{1}{2}d(Ul, l)\right)$$
$$\leq \psi_1\big(d(Ul, l), d(Ul, l), d(Ul, l), d(Ul, l), d(Ul, l), d(Ul, l), d(Ul, l)\big) -$$
$$\psi_2\left(d(Ul, l), \tfrac{1}{2}d(Ul, l), \tfrac{1}{2}d(Ul, l), 0, 0, d(l, Ul), \tfrac{1}{2}d(Ul, l)\right)$$
$$= \varphi_1\big(d(Ul, l)\big) - \psi_2\left(d(Ul, l), \tfrac{1}{2}d(Ul, l), \tfrac{1}{2}d(Ul, l), 0, 0, d(l, Ul), \tfrac{1}{2}d(Ul, l)\right)$$
$$\varphi_1\big(d(Ul, l)\big) < \varphi_1\big(d(Ul, l)\big) \qquad if\ \ Ul \neq l$$

a contradiction.

Therefore, $Ul = l ----(3.1.18)$

Substituting $\quad s = l \;\; and \;\; t = x_{2n-1} \;\; in$ (3.1.1)

$$\varphi_1\big(d(fl, gx_{2n-1})\big) \le \psi_1\begin{pmatrix} d(Ul, Vx_{2n-1}), \frac{1}{2}d(Ul, gx_{2n-1}), \frac{1}{2}d(fl, Vx_{2n-1}), d(Ul, fl), d(Vx_{2n-1}, gx_{2n-1}), \\ \frac{1}{2}[d(gx_{2n-1}, Ul) + d(fl, Vx_{2n-1})], \frac{1}{2}[d(Ul, Vx_{2n-1}) + d(Ul, fl)] \end{pmatrix} - \psi_2\begin{pmatrix} d(Ul, Vx_{2n-1}), \frac{1}{2}d(Ul, gx_{2n-1}), \frac{1}{2}d(fl, Vx_{2n-1}), d(Ul, fl), d(Vx_{2n-1}, gx_{2n-1}), \\ \frac{1}{2}[d(gx_{2n-1}, Ul) + d(fl, Vx_{2n-1})], \frac{1}{2}[d(Ul, Vx_{2n-1}) + d(Ul, fl)] \end{pmatrix}$$

$$\varphi_1\big(d(fl, l)\big) \le \psi_1\begin{pmatrix} d(l, l), \frac{1}{2}d(l, l), \frac{1}{2}d(fl, l), d(l, fl), d(l, l), \\ \frac{1}{2}[d(l, l) + d(fl, l)], \frac{1}{2}[d(l, l) + d(l, fl)] \end{pmatrix} - \psi_2\begin{pmatrix} d(l, l), \frac{1}{2}d(Ul, l), \frac{1}{2}d(fl, l), d(l, fl), d(l, l), \\ \frac{1}{2}[d(l, l) + d(fl, l)], \frac{1}{2}[d(l, l) + d(l, fl)] \end{pmatrix}$$

$$= \psi_1\left(0, 0, \tfrac{1}{2}d(fl, l), d(l, fl), 0, \tfrac{1}{2}d(fl, l), \tfrac{1}{2}d(l, fl)\right) - \psi_2\left(0, 0, \tfrac{1}{2}d(fl, l), d(l, fl), 0, \tfrac{1}{2}d(fl, l), \tfrac{1}{2}d(l, fl)\right)$$

$$\le \psi_1\big(d(fl, l), d(fl, l), d(fl, l), d(fl, l), d(fl, l)\big) - \psi_2\left(0, 0, \tfrac{1}{2}d(fl, l), d(l, fl), 0, \tfrac{1}{2}d(fl, l), \tfrac{1}{2}d(l, fl)\right) = \varphi_1\big(d(fl, l)\big) - \psi_2\left(0, 0, \tfrac{1}{2}d(fl, l), d(l, fl), 0, \tfrac{1}{2}d(fl, l), \tfrac{1}{2}d(l, fl)\right)$$

$$\varphi_1\big(d(fl, l)\big) < \varphi_1\big(d(fl, l)\big) \qquad if\ \ fl \ne l$$

a contradiction.

Therefore, $fl = l$ ----(3.1.19)

$l = fl \in f(X) \subseteq V(X)$ then there exists $h \in X$ such that $l = Vh$ ----(3.1.20)

Substituting $s = x_{2n}$ and $t = h$ in (3.1.1), we get

$$\varphi_1\big(d(fx_{2n}, gh)\big) \le \psi_1\begin{pmatrix} d(Ux_{2n}, Vh), \frac{1}{2}d(Ux_{2n}, gh), \frac{1}{2}d(fx_{2n}, Vh), d(Ux_{2n}, fx_{2n}), d(Vh, gh), \\ \frac{1}{2}[d(gh, Ux_{2n}) + d(fx_{2n}, Vh)], \frac{1}{2}[d(Ux_{2n}, Vh) + d(Ux_{2n}, fx_{2n})] \end{pmatrix} - \psi_2\begin{pmatrix} d(Ux_{2n}, Vh), \frac{1}{2}d(Ux_{2n}, gh), \frac{1}{2}d(fx_{2n}, Vh), d(Ux_{2n}, fx_{2n}), d(Vh, gh), \\ \frac{1}{2}[d(gh, Ux_{2n}) + d(fx_{2n}, Vh)], \frac{1}{2}[d(Ux_{2n}, Vh) + d(Ux_{2n}, fx_{2n})] \end{pmatrix}$$

$From$ (3.1.16), (3.1.17), (3.1.18), (3.1.19), (3.1.20), $we\ get$

$$\varphi_1\big(d(l,gh)\big) \le \psi_1\begin{pmatrix} d(l,l), \frac{1}{2}d(Ul,gh), \frac{1}{2}d(l,l), d(l,l), d(l,gh), \\ \frac{1}{2}[d(gh,l)+d(l,l)], \frac{1}{2}[d(l,l)+d(l,l)] \end{pmatrix}$$

$$-\psi_2\begin{pmatrix} d(l,l), \frac{1}{2}d(Ul,gh), \frac{1}{2}d(l,l), d(l,l), d(l,gh), \\ \frac{1}{2}[d(gh,l)+d(l,l)], \frac{1}{2}[d(l,l)+d(l,l)] \end{pmatrix}$$

$$= \psi_1\left(0, \tfrac{1}{2}d(l,gh), 0, 0, d(l,gh), \tfrac{1}{2}d(gh,l), 0\right) -$$

$$\psi_2\left(0, \tfrac{1}{2}d(l,gh), 0, 0, d(l,gh), \tfrac{1}{2}d(gh,l), 0\right)$$

$$\le \psi_1\big(d(l,gh), d(l,gh), d(l,gh), d(l,gh), d(l,gh), d(l,gh), d(l,gh)\big)$$

$$-\psi_2\left(0, \frac{1}{2}d(l,gh), 0, 0, d(l,gh), \frac{1}{2}d(gh,l), 0\right)$$

$$= \varphi_1\big(d(l,gh)\big) - \psi_2\left(0, \tfrac{1}{2}d(l,gh), 0, 0, d(l,gh), \tfrac{1}{2}d(gh,l), 0\right)$$

$$\varphi_1\big(d(l,gh)\big) < \varphi_1\big(d(l,gh)\big) \qquad if\ \ gh \neq l$$

a contradiction.

Therefore, $gh = l = Vh$ $\qquad ----(3.1.21)$

Therefore, $gVh = Vgh$ $\ (since\ (V,g)\ is\ sub\ compatible)$

$gl = gVh = Vgh = Vl ----(3.1.22)$

Substituting $s = x_{2n}$ and $t = l$ in (3.1.1), we get

$$\varphi_1\big(d(fx_{2n}, gl)\big)$$

$$\le \psi_1\begin{pmatrix} d(Ux_{2n},Vl), \frac{1}{2}d(Ux_{2n},gl), \frac{1}{2}d(fx_{2n},Vl), d(Ux_{2n},fx_{2n}), d(Vl,gl), \\ \frac{1}{2}[d(gl,Ux_{2n})+d(fx_{2n},Vl)], \frac{1}{2}[d(Ux_{2n},Vl)+d(Ux_{2n},fx_{2n})] \end{pmatrix}$$

$$-\psi_2\begin{pmatrix} d(Ux_{2n},Vl), \frac{1}{2}d(Ux_{2n},gl), \frac{1}{2}d(fx_{2n},Vl), d(Ux_{2n},fx_{2n}), d(Vl,gl), \\ \frac{1}{2}[d(gl,Ux_{2n})+d(fx_{2n},Vl)], \frac{1}{2}[d(Ux_{2n},Vl)+d(Ux_{2n},fx_{2n})] \end{pmatrix}$$

$$\varphi_1\big(d(l,gl)\big) \le \psi_1\begin{pmatrix} d(l,gl), \frac{1}{2}d(l,gl), \frac{1}{2}d(l,l), d(l,l), d(gl,gl), \\ \frac{1}{2}[d(gl,l)+d(l,gl)], \frac{1}{2}[d(l,gl)+d(l,l)] \end{pmatrix}$$

$$-\psi_2\begin{pmatrix} d(l,gl), \frac{1}{2}d(l,gl), \frac{1}{2}d(l,l), d(l,l), d(gl,gl), \\ \frac{1}{2}[d(gl,l)+d(l,gl)], \frac{1}{2}[d(l,gl)+d(l,l)] \end{pmatrix}$$

$$= \psi_1\left(d(l,gl), \tfrac{1}{2}d(l,gl), 0, 0, 0, d(gl,l), \tfrac{1}{2}d(gl,l)\right) -$$

$$\psi_2\left(d(l,gl), \tfrac{1}{2}d(l,gl), 0, 0, 0, d(gl,l), \tfrac{1}{2}d(gl,l)\right)$$

$$\leq \psi_1(d(l,gl),d(l,gl),d(l,gl),d(l,gl),d(l,gl)) - \psi_2\left(d(l,gl),\tfrac{1}{2}d(l,gl),0,0,0,d(gl,l),\tfrac{1}{2}d(gl,l)\right)$$

$$= \varphi_1(d(l,gl)) - \psi_2\left(d(l,gl),\frac{1}{2}d(l,gl),0,0,0,d(gl,l),\frac{1}{2}d(gl,l)\right)$$

$$\varphi_1(d(l,gl)) < \varphi_1(d(l,gl)) \qquad if\ \ gl \neq l$$

a contradiction.

Therefore, $gl = l \quad ----(3.1.23)$

Therefore, $gl = Vl = l ----(3.5.24)$

$From$ (3.1.18), (3.1.19) and (3.1.24), $we\ get$

$f, g, U\ and\ V$ have a common fixed point in X.

If $f, g\ and\ V$ is continuous, similarly we can prove $f, g, U\ and\ V$ have a common fixed point in X.

Suppose, $l\ and\ h$ are common fixed point of $f, g, U\ and\ V$.

From (3.1.1),

$$\varphi_1(d(fl,gh)) \leq \psi_1\begin{pmatrix} d(Ul,Vh),\frac{1}{2}d(Ul,gh),\frac{1}{2}d(fl,Vh),d(Ul,fl),d(Vh,gh), \\ \frac{1}{2}[d(gh,Ul)+d(fl,Vh)],\frac{1}{2}[d(Ul,Vh)+d(Ul,fl)] \end{pmatrix}$$
$$- \psi_2\begin{pmatrix} d(Ul,Vh),\frac{1}{2}d(Ul,gh),\frac{1}{2}d(fl,Vh),d(Ul,fl),d(Vh,gh), \\ \frac{1}{2}[d(gh,Ul)+d(fl,Vh)],\frac{1}{2}[d(Ul,Vh)+d(Ul,fl)] \end{pmatrix}$$

$$\varphi_1(d(l,h)) \leq \psi_1\begin{pmatrix} d(l,h),\frac{1}{2}d(l,h),\frac{1}{2}d(l,h),d(l,l),d(h,h), \\ \frac{1}{2}[d(h,l)+d(l,h)],\frac{1}{2}[d(l,h)+d(l,l)] \end{pmatrix}$$
$$- \psi_2\begin{pmatrix} d(l,h),\frac{1}{2}d(l,h),\frac{1}{2}d(l,h),d(l,l),d(h,h), \\ \frac{1}{2}[d(h,l)+d(l,h)],\frac{1}{2}[d(l,h)+d(l,l)] \end{pmatrix}$$

$$= \psi_1\left(d(l,h),\frac{1}{2}d(l,h),\frac{1}{2}d(l,h),0,0,d(h,l),\frac{1}{2}d(l,h)\right)$$
$$- \psi_2\left(d(l,h),\frac{1}{2}d(l,h),\frac{1}{2}d(l,h),0,0,d(h,l),\frac{1}{2}d(l,h)\right)$$

$$\varphi_1(d(l,h)) \leq \psi_1(d(l,h),d(l,h),d(l,h),d(h,l),d(l,h))$$
$$- \psi_2\left(d(l,h),\frac{1}{2}d(l,h),\frac{1}{2}d(l,h),0,0,d(h,l),\frac{1}{2}d(l,h)\right)$$

$$= \varphi_1(d(l,h)) - \psi_2\left(d(l,h),\tfrac{1}{2}d(l,h),\tfrac{1}{2}d(l,h),0,0,d(h,l),\tfrac{1}{2}d(l,h)\right)$$

$$\varphi_1(d(l,h)) < \varphi_1(d(l,h)) \qquad if\ \ h \neq l$$

a contradiction.

Therefore, $h = l$.

Hence, f, g, U and V have a unique common fixed point in X.

4. An application to integral type inequalities:

The following is an application of **Theorem 3.1** to integrals.

4.1 Example: Suppose (X, d) is a complete metric space and f, g, U and V be four mappings from X to itself and satisfying the following integral inequality for all $x, y \in X$ such that

$$\int_0^{\varphi_1(d(fx,gy))} \eta(t)\,dt$$

$$\leq \int_0^{\psi_1\left(d(Ux,Vy),\frac{1}{2}d(Ux,gy),\frac{1}{2}d(fx,Vy),d(Ux,fx),d(Vy,gy),\frac{1}{2}[d(gy,Ux)+d(fx,Vy)],\frac{1}{2}[d(Ux,Vy)+d(Ux,fx)]\right)} \eta(t)\,dt$$

$$- \int_0^{\psi_2\left(d(Ux,Vy),\frac{1}{2}d(Ux,gy),\frac{1}{2}d(fx,Vy),d(Ux,fx),d(Vy,gy),\frac{1}{2}[d(gy,Ux)+d(fx,Vy)],\frac{1}{2}[d(Ux,Vy)+d(Ux,fx)]\right)} \eta(t)\,dt$$

$$---(4.1.1)$$

where $\psi_1, \psi_2 \in \Psi_7$ with $\varphi_1(\alpha) = \psi_1(\alpha, \alpha, \alpha, \alpha, \alpha, \alpha, \alpha)$, $\alpha \in [0, \infty)$ and $\eta : R^+ \longrightarrow R^+$ is a Lebesgue-integrable mapping , which is non negative, summable, and $\int_0^{\varepsilon} \eta(t)\,dt > 0$ for each $\varepsilon > 0$. Suppose

i) One of the four mappings f, g, U and V is continuous.

ii) (f, U) and (g, V) are sub compatible.

iii) $f(X) \subseteq V(X)$ and $g(X) \subseteq U(X)$.

Then f, g, U and V have a unique common fixed point in X.

Proof: We first show that (3.1.1) holds for f, g, U and V.

Suppose for some $x, y \in X$, (3.1.1) does not hold. Then

$$\varphi_1(d(fx,gy)) > \psi_1\begin{pmatrix} d(Ux,Vy),\frac{1}{2}d(Ux,gy),\frac{1}{2}d(fx,Vy),d(Ux,fx),d(Vy,gy), \\ \frac{1}{2}[d(gy,Ux)+d(fx,Vy)],\frac{1}{2}[d(Ux,Vy)+d(Ux,fx)] \end{pmatrix}$$

$$-\psi_2\begin{pmatrix} d(Ux,Vy),\frac{1}{2}d(Ux,gy),\frac{1}{2}d(fx,Vy),d(Ux,fx),d(Vy,gy), \\ \frac{1}{2}[d(gy,Ux)+d(fx,Vy)],\frac{1}{2}[d(Ux,Vy)+d(Ux,fx)] \end{pmatrix}$$

Write $$\varepsilon = \varphi_1(d(fx,gy)) - \begin{pmatrix} \psi_1\begin{pmatrix} d(Ux,Vy),\frac{1}{2}d(Ux,gy),\frac{1}{2}d(fx,Vy),d(Ux,fx),d(Vy,gy), \\ \frac{1}{2}[d(gy,Ux)+d(fx,Vy)],\frac{1}{2}[d(Ux,Vy)+d(Ux,fx)] \end{pmatrix} \\ -\psi_2\begin{pmatrix} d(Ux,Vy),\frac{1}{2}d(Ux,gy),\frac{1}{2}d(fx,Vy),d(Ux,fx),d(Vy,gy), \\ \frac{1}{2}[d(gy,Ux)+d(fx,Vy)],\frac{1}{2}[d(Ux,Vy)+d(Ux,fx)] \end{pmatrix} \end{pmatrix}$$

Then $\varepsilon > 0$.

By hypothesis, $\int_0^{\varepsilon} \eta(t)\, dt > 0$.

Therefore,

$$\int_0^{\varphi_1(d(fx,gy))} \eta(t)\, dt$$

$$> \int_0^{\psi_1\left(d(Ux,Vy),\frac{1}{2}d(Ux,gy),\frac{1}{2}d(fx,Vy),d(Ux,fx),d(Vy,gy),\frac{1}{2}[d(gy,Ux)+d(fx,Vy)],\frac{1}{2}[d(Ux,Vy)+d(Ux,fx)]\right)} \eta(t)\, dt$$

$$- \int_0^{\psi_2\left(d(Ux,Vy),\frac{1}{2}d(Ux,gy),\frac{1}{2}d(fx,Vy),d(Ux,fx),d(Vy,gy),\frac{1}{2}[d(gy,Ux)+d(fx,Vy)],\frac{1}{2}[d(Ux,Vy)+d(Ux,fx)]\right)} \eta(t)\, dt$$

a contradiction.

Therefore,

$$\varphi_1(d(fx,gy)) \le \psi_1\begin{pmatrix} d(Ux,Vy),\frac{1}{2}d(Ux,gy),\frac{1}{2}d(fx,Vy),d(Ux,fx),d(Vy,gy), \\ \frac{1}{2}[d(gy,Ux)+d(fx,Vy)],\frac{1}{2}[d(Ux,Vy)+d(Ux,fx)] \end{pmatrix}$$

$$-\psi_2\begin{pmatrix} d(Ux,Vy),\frac{1}{2}d(Ux,gy),\frac{1}{2}d(fx,Vy),d(Ux,fx),d(Vy,gy), \\ \frac{1}{2}[d(gy,Ux)+d(fx,Vy)],\frac{1}{2}[d(Ux,Vy)+d(Ux,fx)] \end{pmatrix}$$

Thus, (3.1.1) holds for $f, g, U \text{ and } V$.

Therefore, by **Theorem 3.1**, $f, g, U \text{ and } V$ have a unique common fixed point.

The following example also supports our **Theorem 3.1.**

4.2 Example: Suppose (X, d) be a complete metric space with the metric $d(s,t) = \frac{1}{2}|s-t|$ in the interval $X = [0,1]$.

Define $f, g, U \text{ and } V$ are four mapping from X to itself such that

$$fx = \frac{x}{4}\ ,\ gy = \frac{y}{4}\ ,\ Ux = x \text{ and } Vy = y.$$

Let $\psi_1(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\alpha_7) = \max\{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\alpha_7\}$, $\psi_2 = \frac{1}{4}\psi_1$ and $\varphi_1(\alpha) = \alpha$, $\forall\, \alpha \in [0,\infty)$ where $\psi_1, \psi_2 \in \Psi_7$

Then $f, g, U \text{ and } V$ is satisfy the conditions of **Theorem 3.1**.

Therefore, $f, g, U \text{ and } V$ have a unique common fixed point.

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